Math 10B with Professor Stankova
Worksheet, Discussion #4; Tuesday, 1/30/2018
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Binomial Identities

Examples

1. Show that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

Solution: First, we can do a combinatorial reasoning. Suppose that I have n people and I want to choose an r person team and inside that team, a k person special committee. One way to count the number of ways to do this is first I choose the r person team, and I can do this $\binom{n}{r}$ ways. Then, from within this team, I choose the committee and this can occur in $\binom{r}{k}$ ways. Thus, this gives a total of $\binom{n}{r}\binom{r}{k}$ ways.

Another way to do this is first choose the committee and this can occur in $\binom{n}{k}$ ways. After I choose the committee, I can choose the remaining people on the team and there are $r-k$ people left to choose. But now there are only $n-k$ people left to choose from since we chose k of them for the committee. Thus, we can complete the team in $\binom{n-k}{r-k}$ ways. This gives a total of $\binom{n}{k}\binom{n-k}{r-k}$ different ways to do this. Since these are two ways of counting the same number of ways, they are equal and

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Algebraically, we have

$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!}\frac{r!}{k!(r-k)!} = \frac{n!}{(n-r)!k!(r-k)!}.$$

And the other term is

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-k-(r-k))!} = \left(\frac{n!}{k!(r-k)!(n-r)!}\right) = \binom{n}{r}\binom{r}{k}.$$

2. Prove that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

Solution: The binomial theorem tells us that $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. Let $x=1$, then we get the desired equality.

Problems

3. Prove that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

Solution: Plug in $x = 2$ to the binomial theorem $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

4. What is the coefficient of x^2y^3 in $(2x - 3y)^5$?

Solution: The coefficient is $\binom{5}{2}(2x)^2(-3y)^3 = 10 \cdot 2^2 \cdot (-3)^3 = -1080$.

5. (Challenge) What is the coefficient of $x^2y^2z^2$ in $(x + y + z)^6$?

Solution: We can write this as $(x + (y + z))^6$ and so for the term with x^2, we have $\binom{6}{2}x^2(y + z)^4$. The term with y^2z^2 in $(y + z)^4$ is $\binom{4}{2}y^2z^2$ so the term with $x^2y^2z^2$ is $\binom{6}{2}\binom{4}{2} = 90$.

Distinguishable Boxes

Examples

6. How many different Yahtzee rolls are there (rolls are 5 die)?

Solution: Here the balls are the outcome of the die rolls and the urns are the different outcomes, namely 1 through 6. We care about the different outcomes but not about the order of the die so this is indistinguishable balls and distinguishable boxes. This can occur in $\binom{5+6-1}{5} = \binom{10}{5}$ ways.

7. How many four digit increasing numbers are there (1223 is an example)?

Solution: Here, we want to choose the numbers 1 through 9 with repetition 4 different times. The number of ways to do this is $\binom{9+4-1}{4} = \binom{12}{4}$ different ways. We can think of this problem as the numbers 1 through 9 being the different boxes and each number we pick is a ball that we put in the urn.

8. How many ways are there to put 7 balls in 3 boxes if each box must have at least one ball?

Solution: There are 3^7 ways to put the 7 balls in 3 boxes. Let A, B, C be the cases where box $1, 2, 3$ are empty respectively. Then the number of ways for each is 2^7 and the intersections happen in a unique way. Thus, the total number of ways is

$$3^7 - 3 \cdot 2^7 + 3.$$

9. How many ways are there to be n balls in k boxes if each box must have at least one ball?

Solution: We try to do the same thing as the previous problem. There are k^n ways to put the n balls in the k boxes. Let $A_1, A_2, \ldots, A_k$ be the cases where box $1, 2, \ldots, k$ are empty respectively. If A_i is true, box i is empty and so there are only $k-1$ possibilities for the balls giving $(k-1)^n$ ways. There are k ways to choose the box that is empty. When considering the pairs, there are now 2 fewer options so $k-2$ possibilities for the balls to go to and hence $(k-2)^n$ different ways. Here, there are $\binom{k}{2}$ ways to choose the two boxes that are empty. When considering intersection of three of these sets, there are $k-3$ possibilities for the balls to go to and $(k-3)^n$ different ways if 3 boxes are guaranteed empty. And, there are $\binom{k}{3}$ different ways to choose the two boxes that are empty. Putting this all together and using PIE which tells us that we need to alternate signs in between, we get that the total number of ways is

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots + (-1)^{k-1}\binom{k}{k-1}(k-(k-1))^n + (-1)^k\binom{k}{k}(k-k)^n.$$